Interim Report

ESTIMATION OF STATE WITH ACCEPTABLE ACCURACY CONSTRAINTS

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INTRODUCTION

The minimum variance or Kalman formulation for estimating a state vector from observations has been used by numerous investigators. Numerical problems with this technique have appeared in some applications. Those applications which tend to give difficulty are characterized by a sequence of observations which are very sensitive to some elements of the state vector but have little sensitivity to the remaining elements. Another way of stating the condition is that numerically, the measurements do not span the state space.

A typical example of where problems might exist is in the determination of the orbit of a lunar orbiter from Earth-based range and/or range rate tracking data. The numerical problems do not necessarily always occur, but they are likely to occur if tracking data proceeds indefinitely. Numerical difficulty in an orbit determination program is characterized by a gradual increase in the residual magnitudes followed eventually by total loss of the state estimate. Error analysis programs show signs of difficulty by calculating negative variances, that is, the covariance matrix gradually becomes singular. In addition, critics of the method have stated that, because many observations are included, the covariance matrix becomes too small and the latest observations are not sufficiently weighted. Since all error sources are not accounted for, the result is that the estimate of state will not follow the observations as well as one would like.

This report discusses a new technique which should eliminate these problems with the minimum variance estimator. The method to be discussed is based on the philosophy that it is impossible to obtain a perfect estimate of the state vector, regardless of the number and quality of the observations. Instead, the observations are used in a near-optimal manner until the error in the estimate of the state is equal to some acceptable accuracy bound.

Thereafter the observations are continually processed to keep the estimate within this acceptable accuracy bound.

REVIEW OF THE MINIMUM VARIANCE SOLUTION

The minimum variance solution is generally given in the form of a set of recursive equations (for example see Ref. 1).

Time Updating

$$\hat{X}(t) = \int_{t_o}^{t} \dot{\hat{X}}(t)\, dt \tag{1}$$

$$P(t) = \varphi(t;t_o)\, P(t_o)\, \varphi^T(t;t_o) \tag{2}$$

Measurement Updating

$$\hat{X}_n = \hat{X} + PH^T (HPH^T + Q)^{-1} (Y - \hat{Y}) \tag{3}$$

$$P_n = P - PH^T (HPH^T + Q)^{-1} HP \tag{4}$$

$$Y(t) = G(X, t) + q \tag{5}$$

$$\hat{Y}(t) = G(\hat{X}, t) \tag{6}$$

Equation (1) simply says that the estimate of the state, $\hat{X}$, at any time, t, is found by integrating the equations of motion from time, t_o, to time, t. The equations of motion are of the general form

$$\dot{X} = F(X, t) \tag{7}$$

So if we let

$$A(t) = \nabla_X F \Big|_{X(t) = \hat{X}(t)}$$

then for small deviations from $\hat{X}$, given by $\hat{x}$ we have

$$\hat{x}(t) = \varphi(t;t_o)\, \hat{x}(t_o) \tag{8}$$

where

$$\dot{\varphi}(t) = A(t)\ \varphi(t) \qquad \varphi(t_o;t_o) = I \tag{9}$$

Hence, the covariance matrix of errors in the estimate $E\left[(X-\hat{X})(X-\hat{X})^T\right] = P(t)$ is given by eq. (2).

Equation (2) implies perfect modeling of the dynamical system and that all error sources, random as well as deterministic, are modeled. More will be said about this later.

In eq. (3), $Y(t)$ is the observation which is assumed to be a function of the state, given by eq. (5). The computed observation, $\hat{Y}$, is determined from the estimated state, $\hat{X}$, as shown in eq. (6). The other quantities given in eqs. (3)-(5) are

q = random error in measurement

where

$$E(q) = 0$$

$$E(qq^T) = Q$$

$$H = \nabla_X G \Big|_{X = \hat{X}}$$

The use of eq. (3) implies a perfect model for the observation function and all error sources must be modeled.

Discussion of the Minimum Variance Solution

The solution given by eqs. (1)-(6) can be proved to be optimal in the least-squares sense for the case of linear equations. For nonlinear problems no solution is known. However, experience has shown that the above equations work quite adequately for reasonable starting estimates of $\hat{X}$. The problems discussed here arise after calculation of orbits using a large number of observations, wherein the estimate of the state has been good and then drifts away.

Hence, these problems are not linearity difficulties where convergence failure is a characteristic trait. The problems consist of numerical and modeling errors which prevent the filter from maintaining good estimates. Improper modeling of the dynamic and measurement equations, as well as neglecting the effects of numerical computation error, are the fundamental causes.

One obvious cure would be to correct the mathematical models, including the modeling for numerical truncation error effects. Such a cure is not practical for a number of reasons, such as inadequate knowledge, computer size, calculation times, etc. Hence, we need search for a simpler treatment which will give satisfactory results that are less dependent on the completeness of mathematical model and numerical accuracy of the calculations.

Inherent Numerical Difficulties with the Minimum Variance Solution

Before discussing the proposed solution, it is relevant to discuss the fundamental sources of difficulty. The problem will arise in eq. (3) where the measurement, Y, changes the state estimate. Let us examine this equation. Let

$$\Delta X = PH^T (HPH^T + Q)^{-1} (Y-\hat{Y}) \tag{10}$$

If we let Y be a scalar, then the quantity $HPH^T + Q$ is a scalar. Hence, the inverse is not a problem. Suppose we let

$$C(K) = \sum_{I=1}^{N} P(K, I)\, H(I) \tag{11}$$

and

$$B(K) = \sum_{I=1}^{N} |P(K, I)\ H(I)| \tag{12}$$

The vector quantity C, when multiplied by the residual $Y-\hat{Y}$ and divided by the scalar HPH^T+Q, is the quantity ΔX. The elements of ΔX must retain numerical significance if the observations are to improve the estimate of the state.

The vector B contains the sum of the absolute values of the elements of the product PH^T which have been added to find the resultant vector C. It should be reasonably obvious that if we choose a sufficiently small positive number ϵ and find that

$$|C(K)| \leq \epsilon\, B(K) \tag{13}$$

then $C(K)$ has no significance. For the IBM 7094 with single precision $\epsilon \approx 10^{-8}$. If one lets

$$\rho(K) = \frac{|C(K)|}{B(K)}$$

then the exponent of $\rho(K)$ is an indication of its numerical significance in the calculation of $\Delta X(K)$. Certainly if $\Delta X(K)$ has no numerical significance, the K^{th} state estimate based on the PH^T calculation should not be changed.

Most all the numerical difficulties encountered might be attributed to the loss of significance in the covariance matrix P. This is true to some extent so we should examine the operation indicated by eqs. (2) and (4). Consider first eq. (2).

$$P(t) = \varphi\, P(t_o)\, \varphi^T \tag{2}$$

The IBM 7094 truncates rather than rounds off when performing the calculations. Let us assume for simplicity that φ and P are in single precision and that P (t) is computed in double precision and then stored (truncated) in single precision. Then for eq. (2)

$$\left.\begin{aligned} P_{ii}(t) &= \sigma_i^2 (1 - \epsilon_i) \\ P_{ij}(t) &= \sigma_i \sigma_j \rho_{ij} (1 - \epsilon_{ij}) \end{aligned}\right\} \tag{14}$$

The quantities ϵ_i and ϵ_{ij} are truncation errors. Hence following the operation, the computed correlation coefficient ρ_{ij}^c will be in error by

$$\rho_{ij}^c = \rho_{ij}^t \quad \frac{1 - \epsilon_{ij}}{\sqrt{(1 - \epsilon_i)(1 - \epsilon_j)}} \tag{15}$$

The quantity p_{ij}^t means the true value to the significance of the original φ and P .

Equation (15) does not imply any significant change in the correlation coefficients due to numerical difficulties. Hence, it is not believed that the numerical calculation implied by eq. (2) is fundamental to the source of the problem. A means of insurance which has worked favorably in two error analysis programs (Refs. 2 and 3) is to follow the numerical calculation by a multiplication of the diagonal terms of P (t) by the factor $1 + \epsilon$. For the 7094, a value of ϵ of about 10^{-8} has been reasonably adequate.* The MINVAR orbit determination program (Ref. 4) also uses a modified version of this correction technique.

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* Prior to the technique discussed in this report, this was the method used for eliminating covariance matrix difficulties in the error analysis programs described in the references.

Consider eq. (4), repeated here for convenience

$$P_n = P - PH^T (HPH^T + Q)^{-1} HP \tag{4}$$

It has already been shown that some elements of the quantity PH^T may have no numerical significance. If the numerical computations indicated by eq. (4) are carried out, the problem is further aggravated by subtracting a scaled outer product which can lose significance. Note that PH^T is a vector for a single observation, hence, the operation of eq. (4) must cause the eigenvalues of P_n to be smaller than those of P . Without some precautionary procedure for forcing the covariance matrix, P_n , to remain positive semi-definite following the the application of eq. (4), numerical difficulties are to be expected. The problem would exist whether single or double precision arithmetic were used in the calculations, although it should present less of a problem with double precision calculations and storage.

The logic for the calculations required by eq. (13) is considered next. Failure to pass the indicated test would result in $C(K)$ being set to zero. This procedure should help to prevent numerical difficulties with the $P(t)$ matrix and also prevent calculation of fictitious $\Delta X(K)$'s . It cannot, however, restore some overall numerical significance to the $P(t)$ matrix; that is, force it to retain a value commensurate with the actual covariance matrix of error in the estimate of the state.

ESTIMATION OF STATE WITH ACCEPTABLE ACCURACY CONSTRAINTS

The problems previously discussed resulted from 1) numerical errors and 2) errors due to improper mathematical models. The basic theory was derived without either of these problems being considered. Hence, it may be fruitful to reconsider the theoretical assumptions in the light of these problems.

Suppose we define a problem as follows:

Given: $\hat{x}(t_o)$ and $E(x-\hat{x})(x-\hat{x})^T = P(t_o)$

$$x(t) = \varphi\, x(t_o) \ .$$

A sequence of observations

$$y_i(t_i) = h_i(t)\, x(t) + q$$

where q is a random variable with zero mean and variance Q .

Find: An estimate of the state $\hat{x}_{ni}(t_i)$

$\hat{y}_{ni} = h_i\, \hat{x}_n(t_i)$ such that

$E(x-\hat{x}_{ni})^T(x-\hat{x}_{ni})$ is a minimum if (16)

$E(y-\hat{y}_{ni})^2 > \epsilon$

The symbol $\hat{x}_{ni}$ means the estimate of the state including the i^{th} observation.

The conditions imposed by eq. (16) lead to the normal formulation given by eq. (3). Suppose that the indicated value of

$$E\,(y-\hat{y}_{ni})^2 < \epsilon \tag{17}$$

by use of the optimal filter, then the minimization constraint on $E\,(x-\hat{x}_n)^T(x-\hat{x}_n)$ is removed. A new weighting matrix for including the measurements is then computed such that following the observation

$$E\,(y-\hat{y}_{ni})^2 = \epsilon \tag{18}$$

Before going further, one solution of the problem using this additional freedom will be shown. This solution will not have as many numerical problems as the normal minimum variance estimator.

Let

$$\hat{X}_n = \hat{X} + (b\,I)\,PH^T(HPH^T+Q)^{-1}(Y-\hat{Y}) \tag{19}$$

The quantity, b , of eq. (19) is a scalar whose value is unity for the optimum filter. The covariance matrix P_n is then given by

$$P_n = P - (2b-b^2)\,PH^T(HPH^T+Q)^{-1}HP \tag{20}$$

The variance between a perfect measurement and the computed measurement is, for a single observation

$$HP_nH^T = (HPH^T) - (2b-b^2)\,(HPH^T)^2\,(HPH^T+Q)^{-1} \tag{21}$$

To find the scalar gain, b, set $HP_nH^T = \epsilon$ in eq. (21) and solve the quadratic equation, hence

$$b = 1 \pm \sqrt{\frac{\epsilon(HPH^T + Q) - Q(HPH^T)}{(HPH^T)^2}} \tag{22}$$

Real solutions exist when

$$\epsilon \geq Q \left[\frac{HPH^T}{(HPH^T + Q)} \right]$$

If $\epsilon = KQ$, then b of eq. (22) is real when

$$K \geq \frac{\text{(Variance of residual for a perfect measurement)}}{\text{(Variance of residual for the real measurement)}}$$

It is readily shown that real solutions exist only if $HP_nH^T < \epsilon$. Hence, we use $b = 1$ for the imaginary solution and b given by eq. (22) for the real solution. The positive sign for the radical of eq. (22) is taken so that b remains positive.

The fact that such a philosophy reduces the number of numerical problems is evidenced by examining eq. (20).

for $1 < b < 2$	a smaller than optimum outer product is subtracted from P. Hence, P_n will have larger eigenvalues than the optimum $(b = 1)$.
for $b = 2$	$P_n = P$
for $b > 2$	A positive outer product vector is added. This will make the eigenvalues of P_n greater than those of P.

The first solution has the desired character of reducing the numerical difficulties encountered in covariance matrix computations. This indicates that, by redefining the problem in the manner shown in eq. (16)-(18), that is, by introducing acceptable accuracy tolerance constraints, the estimate of the state is improved. All the properties desired are not contained in this first example solution. For instance, if P is set to zero, nothing can ever cause the measurement residuals to affect the estimate of the state $\hat{X}_n$ and furthermore, there is nothing to combat the fact that PH^T may lose numerical significance.

An Alternate Solution

Previous information whose weight is stored in $P(t)$, is not entirely reliable due to modeling and numerical errors. It is proposed that the desired solution should contain the following properties:

1. Weights for current observations should be chosen so that the corresponding changes in the estimate of the state will reflect the quality of the current observations.

2. The upper bound on acceptable accuracies should be approached from either direction in an asymptotic fashion. For example, if $P = 0$, $P(t)$ should grow until $HP_nH^T = \epsilon$. Similarly, if $HP_nH^T > \epsilon$, the filter should give $HP_nH^T = \epsilon$ only in the limit of an infinite number of observations.

These two properties govern the solution so that it is never optimal in the normal sense.

In accordance with property No. 1, the direction of the change in the state vector ΔX will be chosen as the sum of the following two components ΔX_1 and ΔX_2 :

1. If the past information can be completely relied upon

$$\Delta X = \Delta X_1 = PH^T (HPH^T + Q)^{-1} \tag{23}$$

2. If there were no a priori information ($P = \infty$)

$$\Delta X = \Delta X_2 = \epsilon H^T (HPH^T + Q)^{-1} \tag{24}$$

The proof that eq. (24) is an optimal estimate can be found by considering a single observation. Let

$$y = Hx + q$$

Assuming a maximum likelihood formulation with no a priori information, then

$$x = (H^T Q^{-1} H)^{\#} H^T Q^{-1} y \tag{25}$$

(the symbol # denotes the pseudo inverse).

$$(H^T Q^{-1} H)^{\#} = \frac{H^T QH}{\left(HH^T\right)^2} \tag{26}$$

Hence

$$x = \frac{H^T}{HH^T} y \tag{26}$$

Introduction of the scale factor, $\epsilon (HPH^T + Q)^{-1}$ in eq. (24) allows ΔX_2 to be proportional to the optimal value [eq. (26)] for the condition where no a priori knowledge is available.

The new estimate of the state following an observation is

$$\hat{X}_n = \hat{X} + (PH^T + \epsilon H^T)(HPH^T + Q)^{-1}(Y - \hat{Y}) \tag{27}$$

and

$$E(X - \hat{X}_n)(X - \hat{X}_n)^T = P_n \tag{28}$$

$$P_n = P - PH^T (HPH^T + Q)^{-1} HP + \epsilon^2 (HPH^T + Q)^{-1} H^T H$$

The addition of an outer product vector which causes the eigenvalues of P_n to be larger than the values given by the optimum estimator ($\epsilon = 0$) is noted in eq. (28). Updating of $\hat{X}(t)$ and $P(t)$ with respect to time utilizes eqs. (1) and (2). Using eq. (28), the boundary value on HP_nH^T for an infinite number of observations (at the same time point) is found

$$HP_nH^T = \epsilon HH^T \tag{29}$$

For various types of measurements, HP_nH^T has the dimensions of the measurement squared. If Y is a range measurement, then $HH^T = 1$. If ϵ were chosen as KQ, then K will represent that fraction of the variance of an individual measurement which one can expect to determine range, after an infinite number of observations. The value of K to be used is subject to the specific problem where the filtering is applied.

A Simple Example

As previously mentioned, perfect modeling of a dynamical problem is never possible. The filtering equation shown (27) should have merit in problems where errors due to the equations of motion are neglected. A very simple problem has been chosen to illustrate this point.

Actual Model

$$\left.\begin{aligned} \dot{X} &= 0 \cdot X + e \\ Y &= X + q \end{aligned}\right\} \tag{30}$$

Simulated Model

$$\left.\begin{aligned} \dot{\hat{X}} &= 0 \cdot \hat{X} \\ \hat{Y} &= \hat{X} \end{aligned}\right\} \tag{31}$$

The actual model has the solution

$$X(t) = X(0) + e\ t \tag{32}$$

The simulated model has the solution

$$\hat{X}(t) = \hat{X}(0)$$

A problem as simple as this, in any real situation, could only have been formulated by a person with too much education and too little common sense. If a measuring device is available, the instrument readings would be believed rather than trusting estimates from a theoretical model. This theory, using accuracy tolerances, forces the observations to be considered in the problem solution.

Figure (1) shows the estimated results for $\epsilon - 1$ and $X(0) = 10$ for the Kalman, maximum likelihood (or least squares), and the new estimator. For the Kalman and the new estimator, $\hat{X}(0) = 10$ and $P(0+) = Q$. Also for the new estimator, ϵ of eq. (27) is chosen as

$$\epsilon = KQ$$

$K = 1., .5, .0$ are illustrated for the example where $Q = 1$.

Following the observation measurement and using two values of K for the new estimator, the values of the P_n's are computed

$$\text{for} \quad \underline{K = 1.0} \quad P_n = P - \frac{P^2}{P+1} + \frac{1}{P+1} = \frac{P+1}{P+1} = 1 \tag{34}$$

$$\text{for} \quad \underline{K = .5} \quad P_n = P - \frac{P^2}{P+1} + \frac{.25}{P+1} = \frac{P+.25}{P+1} \tag{35}$$

Table 1

K = .5

OBSERVATION TIME	RESIDUAL	$\frac{P+.5}{P+1}$	X(t)	P_n
2	2	.75	11.5	.625
4	2.5	.693	13.23	.538
6	2.77	.67	15.09	.512
8	2.91	.67	17.05	.504
10	2.95	.67	18.97	.500

Figure 1 illustrates the advantage of retaining trust in current observations for a simple case of improper modeling. The Kalman and maximum likelihood estimates are equal and very poor for this example. They also have the characteristic of worsening with time (number of measurements). For $K = 1$, the new estimator simply tracks the measurements. Hence, the error in the estimate of the observation is the error in the observation. For $K = .5$, the estimated state lags the measurement. Examining Table 1 shows, however, that the residuals have very nearly reached a constant value and that P_n and the gain $\frac{PH^T + \epsilon H^T}{(HPH^T + Q)}$ have reached a constant value. Therefore, although the error in the estimate has grown to a value biased from the observation, it will never get worse than this value so long as observations are included in the estimate. For any $0 < K < 1$, the solution will behave similarly. That is, the error will reach a steady-state value rather than an ever-increasing value as is true for the Kalman or maximum likelihood estimators.

A More Realistic Example

Suppose down-range tracking of a launch vehicle whose launch guidance system includes an inertial platform is the problem. Telemetry data from the inertial system is assumed available so that measurements of vehicle attitude and acceleration are given.

The equations of motion for such a system are of the form

$$\ddot{X} = G(X, t) + A_m \tag{36}$$

A_m = the measured acceleration (due to thrust, drag, etc.).

$G(X, t)$ = the gravitational acceleration.

A_m may be considered as being the true acceleration + errors caused by biases in the accelerometers, gyro-drifts, etc.

Suppose the state, X, is to be estimated including down-range tracking data starting from lift off. At lift off the uncertainty in the state is assumed zero. Direct integration of eq. (36) results in errors in the estimate which are caused by the onboard platform measurement error sources. Suppose the down-range tracking data is to be utilized in estimating the state but the complexity of modeling the onboard measurement sources is a deterring factor.

For the Kalman estimator

$$P(t) = \varphi P(t_o) \varphi^T$$

Therefore, if $P(t_o) = 0$ and the onboard measurement error sources are not modeled, then $P(t) = 0$. Hence, tracking data is not included in the estimate.

The new estimator does not, however, require this modeling (except for error analysis procedures). That is, the state estimate will change with successive observations since the weighting matrix P_n grows after each observation.

An error analysis of the inertial system only and of the inertial system augmented by tracking for a sample case where $\epsilon = KQ$ was carried out. The results are shown in Figure 2. The error analysis program used for this purpose is a modified version of the program in Ref. 5. As can be seen from the results, the new estimator does allow an improvement in the state estimate. This occurs even though the system is not properly modeled. Twelve tracking stations were involved in the error analysis of the simulated example. The complete description of all constants is not included here for simplicity and brevity.

The results shown are not at all surprising. Tracking data should be able to update an inertial system without the necessity of modeling all the inertial error sources. The new estimator is simply one means of doing just this.

CONCLUDING REMARKS

A new method for estimating the state of a system from observation data has been described. This method is based upon setting acceptable accuracy constraints on the error in the estimate, thus allowing the derivation of non-optimal filters which have much better numerical implementation characteristics than the Kalman (optimal) formulation. The new filter also has the desired property of keeping the estimate close to the values implied by current observations. It has been shown that this characteristic is very valuable for error reduction in situations where the equations of motion of the state are not properly modeled. In addition, a philosophy such as this permits the use of a simpler filter than what normally may be required in complex situations.

It is difficult to predict what effect the impact of the above described procedure for estimating the state of a system from observations will have on data reduction programs. The method proposed would seem to be most useful for onboard calculation procedures where simplicity, computer size, and speed have a much more significant importance than for programs designed for large, general-purpose Earth-based computers. The new method could, however, remove the numerical difficulties experienced by large, general-purpose computer programs designed in accordance with the minimum variance solution.

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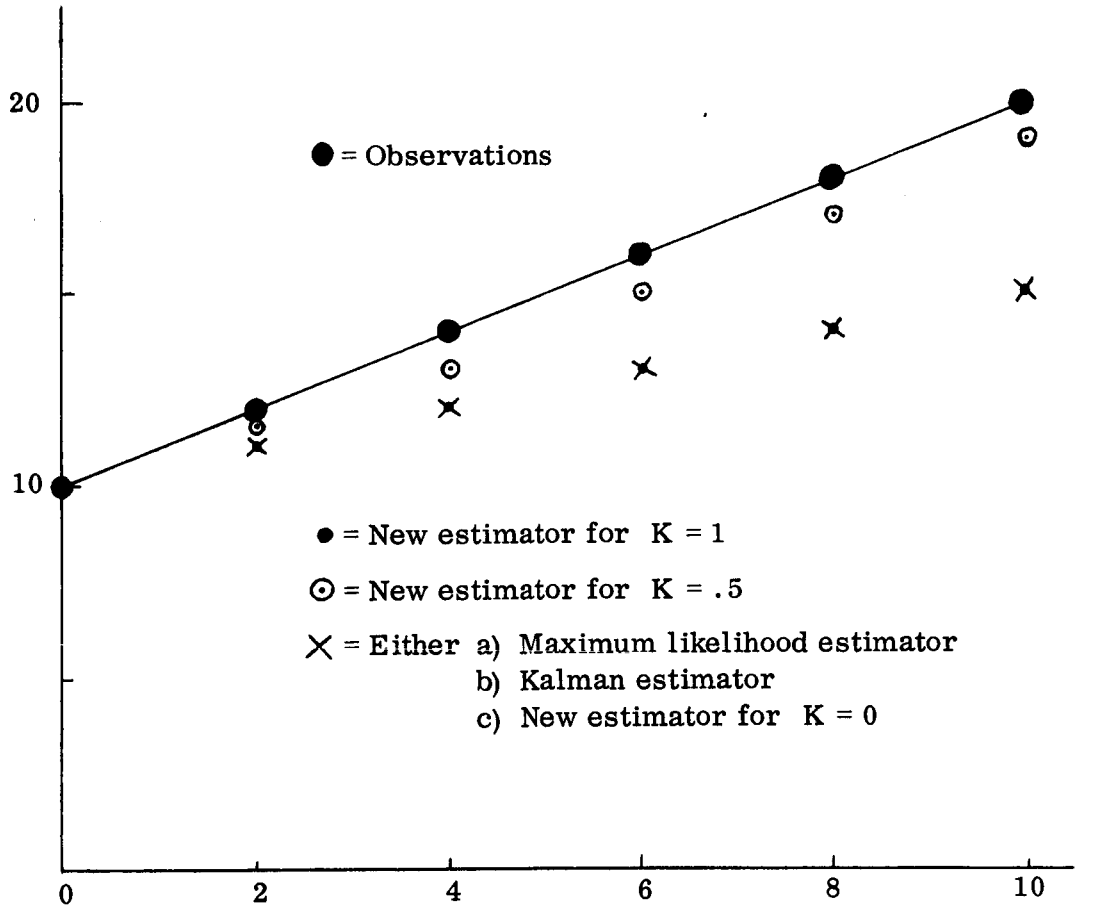
20
10
0
2
4
6
8
10
● = Observations
● = New estimator for K = 1
⊙ = New estimator for K = .5
× = Either a) Maximum likelihood estimator
b) Kalman estimator
c) New estimator for K = 0

Figure 1

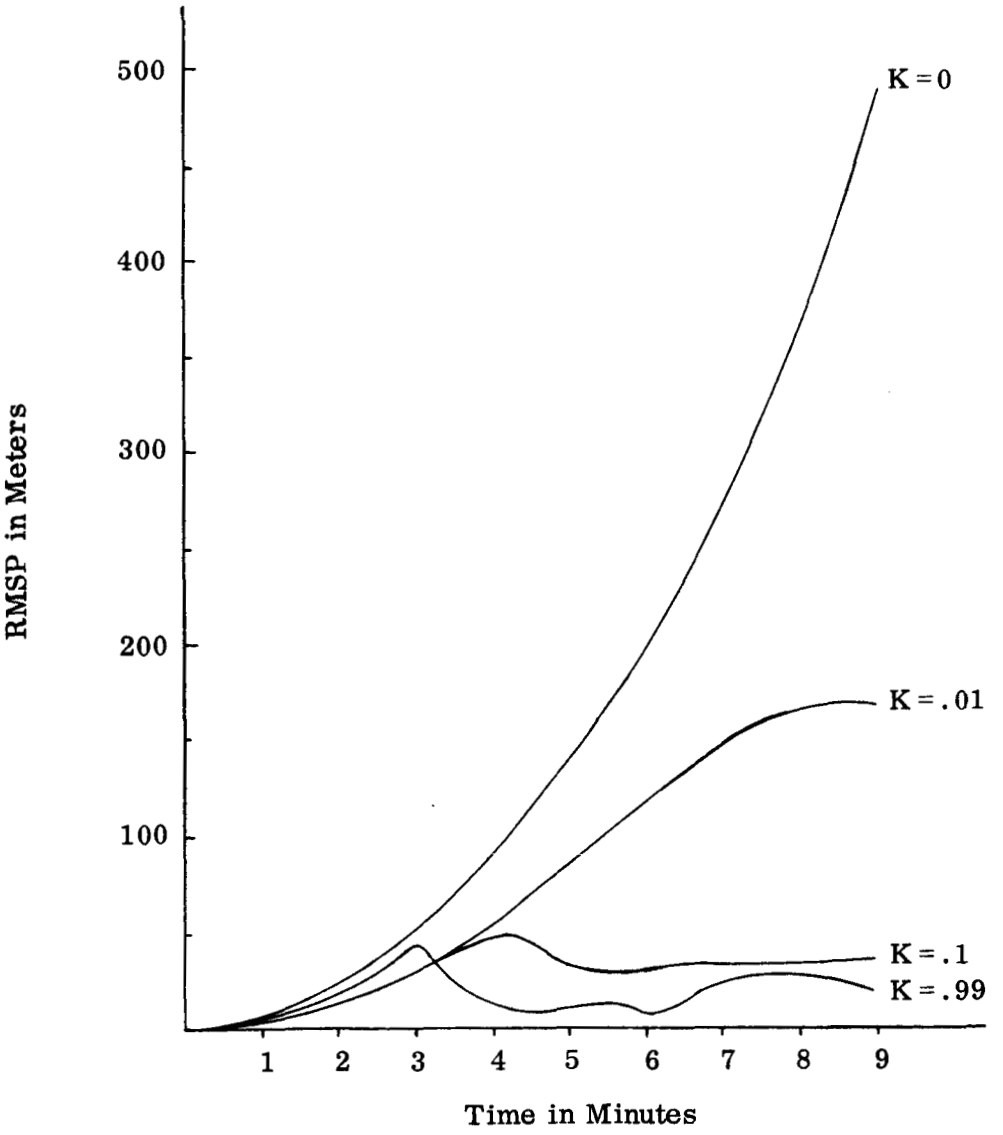

Figure 2A

RMSP = Root Mean Square Position Error